T. Mrowka - 30/5/08 - knot invariants from instantons

<u>Observe</u>: Khovanov homology of unknot $\mathcal{U} \rightsquigarrow \mathbb{Z}^2$

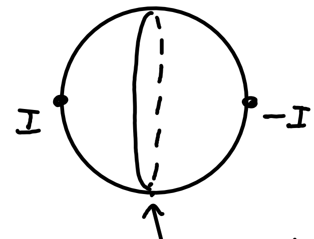
trefoil  $\rightsquigarrow \mathbb{Z}^4 \oplus \mathbb{Z}/2$

Look at representations $\rho : \pi_1(S^3 - K) \to SU(2)$

$\pi_1(S^3 - K)$ is normally generated by any meridian μ (up to conjugacy)

$\rightsquigarrow$ fix conjugacy class in $SU(2)$ for $\rho(\mu)$

conj. classes in $SU(2) \simeq S^3$:



conj. class of $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$\Rightarrow$ require $\rho(\mu) \in \text{conj.}\left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)$

$R(K) :=$ space of such ρ's

<u>Ex.</u> $R(\text{unknot}) \simeq S^2 \rightsquigarrow H_* = \mathbb{Z}^2$

$R(\text{trefoil}) \simeq S^2 \amalg \mathbb{RP}^3 \rightsquigarrow H_* = \mathbb{Z}^4 \oplus \mathbb{Z}/2$!!!

<u>Recall</u>: Y^3 closed manifold $\rightsquigarrow$ $P = Y \times SU(2)$ principal bundle

$A =$ connection in P

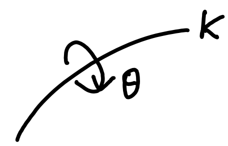
$$CS(a) := \int_Y tr\left(a \wedge da + \frac{1}{3} a \wedge a \wedge a\right)$$

$$d_a CS(b) = \int_Y tr(F_a \wedge b)$$ $\Rightarrow$ critical points are flat conn's., ie. conj classes of representations $\pi_1(Y) \to SU(2)$

<u>Instanton Floer homology</u> = Morse homology for CS

(well defined if Y is a homology sphere; if Y has nontrivial homology, take $P = U(2)$-bundle with a fixed non-trivial det. line)

- Consider connections on $Y \setminus K$ s.t., near K, the connection 1-form is: $a \sim i\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} d\theta$



for some fixed constant α

$\Rightarrow$ holonomy of a around K is $\begin{pmatrix} e^{2\pi i \alpha} & \\ & e^{-2\pi i \alpha} \end{pmatrix}$

Here we'll take $\underline{\alpha = \frac{1}{4}}$

(~ 1993 Kronheimer - M.)

- $CS : \mathcal{A}_\alpha(Y, K) \to \mathbb{R}$
  critical points are conjugacy classes of representations
  $\rho : \pi_1(Y \setminus K) \to SU(2)$ st. $\rho(\mu) \in C(\alpha) =$ conj. class $\begin{bmatrix} e^{2\pi i \alpha} & \\ & e^{-2\pi i \alpha} \end{bmatrix}$ $(\alpha = 1/4)$.

- One issue : want actual space of ρ's, <u>not</u> conjugacy classes ?

<u>Trick</u>: consider on T^3 the flat $SO(3)$-bundle corresponding to direct sum of 3 real line bundles $L_{++-} \oplus L_{+-+} \oplus L_{+--}$

($L_{\pm\pm\pm} \to T^3$ = the 8 real line bundles, with holonomies / generators $= \pm 1$ ...)

ie. $\rho : \pi_1(T^3) \to SO(3)$

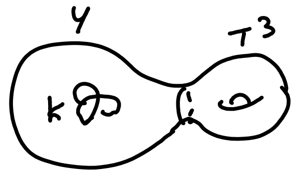
$\gamma_1 \mapsto id$

$\gamma_2 \mapsto diag(1, -1, -1)$

$\gamma_3 \mapsto diag(-1, 1, -1)$

This flat bundle admits exactly 2 lifts to projectively flat $U(2)$-bundles over T^3

$\Rightarrow$ look now at $(Y \# T^3, K)$ instead of (Y, K)



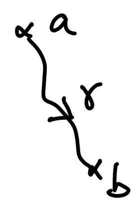
Then $\underline{R(Y \# T^3, K)/_{SU(2)} \cong 2 \text{ copies of } R(Y,K)}$.

$\Rightarrow$ try to do Morse homology for CS on $\mathcal{A}_\alpha(Y \# T^3, K) \to \mathbb{R}$

$\rightsquigarrow$ descends to $\mathcal{A}_\alpha(Y \# T^3, K)/\mathcal{G}_\alpha \to \mathbb{R}/\pi$

★ Monotonicity condition (or: spectral flow and Novikov homology...)

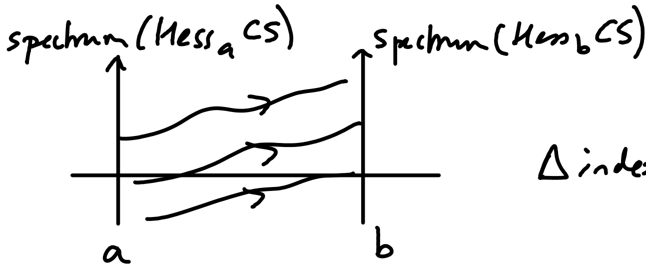
- In Morse-Floer theory, only the relative Morse index of a pair of critical points is defined.



$\Rightarrow$ look at spectral flow of $\mathrm{Hess}_{\gamma(t)}(f)$

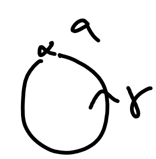
in our case, $f = CS$

$\mathrm{Hess}_a f \sim * d_a$



$\Delta \text{index} = sf(\gamma)$

$\rightsquigarrow$ get in particular for a closed loop



$$\Delta_{sf} : \pi_1(\mathcal{C}) \to \mathbb{Z}$$

| <u>Monotonicity</u> := $\Delta_{sf} : \pi_1(\mathcal{C}) \to \mathbb{Z}$
| $\Delta_{CS} : \pi_1(\mathcal{C}) \to \mathbb{R}$
| $\Delta_{CS} = \lambda \Delta_{sf}, \ \lambda \neq 0$

$\rightsquigarrow$ this is the key technical property.

★ Ordinary instanton case:

$$\pi_1(\mathcal{A}/\mathcal{G}) \simeq \pi_0(\mathcal{G}) \simeq [Y, SU(2)] \simeq \mathbb{Z} \quad \Rightarrow \quad \text{proportionality ok}$$

In our case:

$$\pi_1(\mathcal{A}_\alpha/\mathcal{G}) \simeq \pi_0(\mathcal{G}) \simeq [(Y,K),(SU(2),U(1))] \simeq \mathbb{Z}^2$$

$$(k, \ell)$$

3-dim! degree   1-dim! degree

$$\Delta_{sf}(k,\ell) = 8k + 4\ell$$

$$\Delta_{cs}(k,\ell) = k + 2\alpha\ell \quad \Rightarrow \quad \underline{\text{monotone for } \alpha = \tfrac{1}{4}} \ \checkmark$$

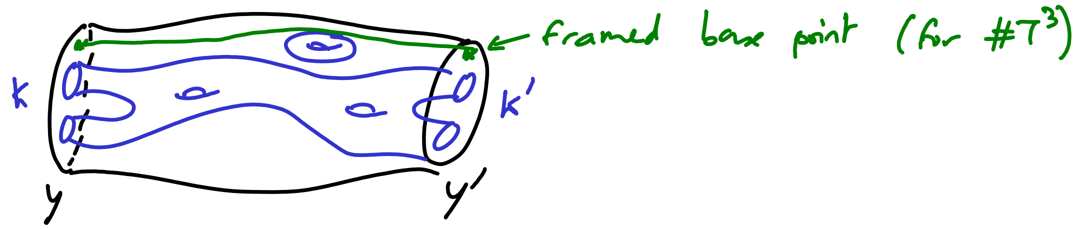
<u>So:</u>

$FI_*(Y,K) :=$ Morse homology of $CS : \mathcal{A}_{\frac{1}{4}}(Y\#T^3, K) \to \mathbb{R}$

$K \subset Y^3$ link

$\mathbb{Z}/4$ - graded theory

★ FI_* has <u>functorial properties</u> wrt <u>cobordisms</u>



<u>$\widetilde{Cob}_{2,1}$</u>: <u>objects</u> $= (Y, L, fr)$

Y = oriented 3-manifold (nonempty, <u>connected</u>)

L = link in Y with a Pin-structure

fr = framed point

<u>morphisms</u> $= (W, \Sigma, fr)$

W = oriented 4-mfld (<u>connected</u>)

Σ = two-mfld (not necess. orientable) w/ pin-structure.

+ with homology orientations

$\Lambda(W)$ for a closed 4-mfld = orientation for $H^0(W)\oplus H^1(W)\oplus H^2_+(W)$
$\Lambda(\Sigma)$ ——"—— 2-mfld = orientation for $H^0(\Sigma)\oplus H^1(\Sigma)\oplus H^2(\Sigma)$

Then get a functor from $\widetilde{Cob}_{2,1}$ ....

- If Pin-structure on Σ is changed: π, π' where $\pi' = \pi \otimes \delta$, $\delta \in H^1(\Sigma, \partial\Sigma; \mathbb{Z}_2)$ $\Rightarrow$ δ^2 is the change

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<u>Conjecture:</u> there is a spectral sequence from $Kh(K) \oplus Kh(K)$ to $FI_*(S^3, K)$.

<u>Model:</u> Ozsvath-Szabó: $Kh^{red}_{\mathbb{Z}_2}(K) \Rightarrow HH(D(K); \mathbb{Z}_2)$

Heegaard-Floer of double branched cover